Europ. J. Combinatorics (1990) **11**, 181–188

Further Combinatorial Properties of Two Fibonacci Lattices

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In an earlier paper on differential posets, two lattices Fib(r) and $Z(r)$ were defined for each positive integer r, and were shown to have some interesting combinatorial properties. In this paper the investigation of Fib(r) and $Z(r)$ is continued. A bijection Ψ: Fib$(r) \to Z(r)$ is shown to preserve many properties of the lattices, though Ψ is not an isomorphism. As a consequence we give an explicit formula which generalizes the rank generating function of Fib(r) and of $Z(r)$. Some additional properties of Fib(r) and $Z(r)$ are developed related to the counting of chains.

1. INTRODUCTION

In [3] two lattices, denoted Fib(r) and $Z(r)$, were defined for each positive integer r and were shown to have some interesting combinatorial properties. (Fib(1) had previously been considered in [1], where it was called the 'Fibonacci lattice'.) In particular, Fib(r) and $Z(r)$ have a unique minimal element $\hat{0}$, are graded, and have the same (finite) number of elements of each rank. When $r = 1$, the number of elements of rank n is the Fibonacci number F_{n+1} (where $F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$). There is a rank-preserving bijection ψ: Fib$(r) \to Z(r)$, which satisfies $e(x) = e(\psi(x))$ for all $x \in$ Fib(r), where $e(x)$ denotes the number of maximal chains in the interval $[\hat{0}, x]$ (see [3, Prop. 5.7]).

In this paper we show that in fact the intervals $[\hat{0}, x]$ and $[\hat{0}, \psi(x)]$ have the same number of chains (or multichains) of any specified length. These numbers are relatively easy to compute for Fib(r), so we have 'transferred' this result to $Z(r)$. As a consequence, we show that for any fixed $n \geq 1$,

$$\sum_{x_1 \leq x_2 \leq \cdots \leq x_n} q^{\rho(x_n)} = \prod_{i=1}^{n} (1 - rq - ((i-1)r + 1)q^2)^{-1},$$

where the sum ranges over all n-element multichains in Fib(r) or in $Z(r)$, and where ρ denotes rank. Our results can also be interpreted in terms of the zeta polynomial [2, Ch. 3.11] of certain subposets of Fib(r) and $Z(r)$. The proof in [3] that $[\hat{0}, x]$ and $[\hat{0}, \psi(x)]$ have the same number of maximal chains does not extend to chains of smaller lengths, so we use a new method of proof here.

We will use the notation

$$\mathbb{N} = \{0, 1, 2, \ldots\}, \qquad \mathbb{P} = \{1, 2, 3, \ldots\}.$$

2. MULTICHAINS IN Fib(r) AND $Z(r)$

We first define the lattices Fib(r) and $Z(r)$. Let $A(r) = \{1_1, 1_2, \ldots, 1_r, 2\}$ be an alphabet with r types of 1's and with one 2. (When $r = 1$ we simply let $A(1) = \{1, 2\}$.) Then Fib(r) and $Z(r)$ have the same set of elements, namely the set $A(r)^*$ of all finite words with letters in $A(r)$ (including the empty word ϕ). The cover relations (and hence by transitivity the entire partial order) of Fib(r) and $Z(r)$ are defined as follows. We say that v covers u in Fib(r) if u is obtained from v by changing a single 2 to a 1_i for some i, or by deleting the last letter in v if it is a 1_i. For instance, the word $v = 221_221_21_1$ in Fib(2) covers the words $1_121_221_21_1$, $1_221_221_21_1$, $21_11_221_21_1$, $21_21_221_21_1$, $221_21_11_21_1$, $221_21_21_21_1$, and 221_221_2. We say that v covers u in $Z(r)$ if u can be obtained

0195-6698/90/020181 + 08 \$02.00/0

from v by changing a single 2 to a 1_i for some i, provided that all letters preceding this 2 are also 2's, or by deleting the first letter which is not a 2 (if it occurs). Thus in $Z(2)$ the word $v=221_221_21_1$ covers the words $1_121_221_21_1$, $1_221_221_21_1$, $21_11_221_21_1$, $21_21_221_21_1$ and 2221_21_1. (Note that v covers 7 words in Fib(r) and 5 in $Z(r)$.)

It is easily seen that Fib(r) and $Z(r)$ are graded posets with $\hat{0}=\phi$ (the empty word), and rank function given by

$$\rho(a_1a_2\cdots a_k)=a_1+a_2+\cdots+a_k,$$

where $a_i\in A(r)$, and where we add the a_i's as integers (ignoring subscripts on the 1's). It is also easily seen [1] [3, after Def. 5.6] that Fib(1) is a distributive lattice, while Fib(r) for any r is upper-semimodular. More strongly, if $x\in$ Fib(r) and x^* is the join of all elements covering x, then the interval $[x, x^*]$ is the product of a boolean algebra with the modular lattice of rank two and cardinality $r+2$. In particular, Fib(2) is 'join-distributive'. (In [3] it was erroneously claimed that Fib(r) is join-distributive for any r.)

We will need the following result from [3, Prop. 5.4]:

Proposition 2.1. *$Z(r)$ is a modular lattice for which every complemented interval has length $\leqslant 2$.*

Given $x\in A(r)^*$ and $n\in\mathbb{P}$, let $M_n(x)=M_n(x,r)$ (respectively, $N_n(x)=N_n(x,r)$) denote the number of multichains $\hat{0}=x_0\leqslant x_1\leqslant\cdots\leqslant x_n=x$ in Fib(r) (respectively, $Z(r)$) of length n with top x. It is clear from the definitions of Fib(r) and $Z(r)$ that if x and x' are two words in $A(r)^*$ differing only in the subscripts on the 1's, then there are automorphisms of Fib(r) and of $Z(r)$ which send x to x'. Hence $M_n(x)=M_n(x')$ and $N_n(x)=N_n(x')$. For this reason we often suppress the subscripts on the 1's in x when writing $M_n(x)$ or $N_n(x)$ for particular x. For instance, $M_n(211y)$ denotes $M_n(21_i1_jy)$ for any $i, j\in\{1,\ldots,r\}$ (and $y\in A(r)^*$).

In the terminology of [2, Ch. 3.11], $M_n(x)$ and $N_n(x)$ are (as functions of n) the *zeta polynomials* of the interval $[\hat{0},x]$ of Fib(r) and $Z(r)$, respectively.

Lemma 2.2. *Let $u\in A(r)^*$. Then*

$$M_n(1u)=\sum_{i=1}^{n}M_i(u),\qquad M_n(2u)=\sum_{i=1}^{n}((i-1)r+1)M_i(u). \tag{1,2}$$

Proof. Let $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant r$. Given a multichain $\hat{0}=u_0\leqslant u_1\leqslant\cdots\leqslant u_i=u$ in Fib(r), associate with it the multichain $\hat{0}=x_0=x_1=\cdots=x_{n-i}<1_ju_1\leqslant\cdots\leqslant 1_ju_i=1_ju$ in Fib(r). This sets up a bijection which proves (1).

Again given $\hat{0}=u_0\leqslant u_1\leqslant\cdots\leqslant u_i=u$ in Fib(r), define the following $(i-1)r+1$ multichains of length n from $\hat{0}$ to $2u$ in Fib(r):

$$\hat{0}=x_0=x_1=\cdots=x_{n-i}<1_ku_1\leqslant 1_ku_2\leqslant\cdots\leqslant 1_ku_s\leqslant 2u_s\leqslant\cdots\leqslant 2u_i=2u,$$
$$1\leqslant k\leqslant r,\ 2\leqslant s\leqslant i,$$
$$\hat{0}=x_0=x_1=\cdots=x_{n-i}<2u_1\leqslant 2u_2\leqslant\cdots\leqslant 2u_i=2u.$$

Every multichain of length n from $\hat{0}$ to $2u$ occurs exactly once in this way, so (2) follows. □

Lemma 2.3. *For any $i\geqslant 0$ and any $u\in A(1)^*$, we have*

$$N_n(2^i1u)-N_{n-1}(2^i1u)=r\sum_{j=1}^{i}N_n(2^{j-1}12^{i-j}1u)+N_n(2^iu)-irN_n(2^{i-1}1u), \tag{3}$$

$$N_n(2^i)-N_{n-1}(2^i)=r\sum_{j=1}^{i}N_n(2^{j-1}12^{i-j})-(ir-1)N_n(2^{i-1}). \tag{4}$$

(*Set $N_n(2^{-1}1u)=0$ and $N_n(2^{-1})=0$ in the case $i=0$.*)

PROOF. Let P be any locally finite poset for which every principal order ideal $\Lambda_x := \{y \in P: y \leqslant x\}$ is finite. Let $L_n(x)$ be the number of multichains $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n = x$ in P. Clearly,

$$L_n(x) = \sum_{y \leqslant x} L_{n-1}(y).$$

Hence, letting μ denote the Möbius function of P we have, by the Möbius inversion formula [2, Prop. 3.7.1],

$$L_{n-1}(x) = \sum_{y \leqslant x} L_n(y)\mu(y, x).$$

Since $\mu(x, x) = 1$ there follows

$$L_n(x) - L_{n-1}(x) = -\sum_{y<x} L_n(y)\mu(y, x). \tag{5}$$

Now, given $x \in Z(r)$, let x_* be the meet of elements which x covers. (Since $Z(r)$ is a lattice by Proposition 2.1, it follows that x_* exists.) By a well known property of Möbius functions (e.g. [2, Cor. 3.9.5]), we have $\mu(y, x) = 0$ unless $x_* \leqslant y \leqslant x$. But by Proposition 2.1, the interval $[x_*, x]$ has length at most 2 (since a finite modular lattice is complemented if and only if $\hat{0}$ is a meet of coatoms).

If $[x_*, x]$ has length 0, then $x = \hat{0}$ and the lemma is clearly valid (put $i = 0$ in (4) to obtain $0 = 0$).

If $[x_*, x]$ has length 1, then $[x_*, x] = [u, 1_j u]$ or $[x_*, x] = [1, 2]$; the latter case only for $r = 1$ (so $j = 1$). Then $\mu(x_*, x) = -1$, and equations (3) (with $i = 0$) and (4) (with $i = r = 1$) coincide with (5).

Finally, assume that $[x_*, x]$ has length 2. If x covers k elements y, then $\mu(y, x) = -1$, and $\mu(x_*, x) = k - 1$. Now if $x = 2^i 1_k u$ (with $i \geqslant 1$) then x covers the $ir + 1$ elements $y = 2^{j-1}1_m 2^{i-j} 1_k u$ ($1 \leqslant j \leqslant i$ and $1 \leqslant m \leqslant r$) or $y = 2^i u$; and $x_* = 2^{i-1}1_k u$. If $x = 2^i$ (with $i > 0$, and with $i > 1$ if $r = 1$) then x covers the ir elements $y = 2^{j-1}1_k 2^{i-j}$ ($1 \leqslant j \leqslant i$, $1 \leqslant k \leqslant r$); and $x_* = 2^{i-1}$. Thus equations (3) and (4) again coincide with (5), and the proof is complete. □

We come to the main result of this section.

THEOREM 2.4. *For all $w \in A(1)^*$ and $n \geqslant 1$, we have $M_n(w, r) = N_n(w, r)$. (Recall that in the notation $M_n(w, r)$ and $N_n(w, r)$, w stands for any word $w' \in A(r)^*$ obtained from w by replacing each* 1 *with some 1_i for $1 \leqslant i \leqslant r$.)*

PROOF. Given a function $F: \mathbb{P} \rightarrow \mathbb{Z}$, define new functions σF and τF by

$$\sigma F(n) = \sum_{i=1}^{n} F(i), \qquad \tau F(n) = ((n-1)r + 1)F(n).$$

If $w = w_1 w_2 \cdots w_k \in A(1)^*$, define the operator Γ_w on functions $F: \mathbb{P} \rightarrow \mathbb{Z}$ by replacing each 1 in w with σ and each 2 with $\sigma\tau$. For instance, $\Gamma_{22121} = \sigma\tau\sigma\tau\sigma\sigma\tau\sigma$. Let $I: \mathbb{P} \rightarrow \mathbb{Z}$ be defined by $I(n) = 1$ for all n. Then it follows from Lemma 2.2 and the initial condition $M_n(\phi) = 1$ that

$$M_n(w) = \Gamma_w I(n). \tag{6}$$

Hence (since clearly $N_n(\phi) = 1$) it suffices to show that the right-hand side of (6) satisfies the same recurrence, given by Lemma 2.3, that $N_n(w)$ satisfies.

We claim that the operators σ and τ satisfy the relation

$$r\sigma^2 = \tau\sigma - \sigma\tau + r\sigma; \tag{7}$$

for we have

$$r\sigma^2F(n)=r\sum_{i=1}^{n}(n-i+1)F(i),\qquad \tau\sigma F(n)=((n-1)r+1)\sum_{i=1}^{n}F(i),$$

$$\sigma\tau F(n)=\sum_{i=1}^{n}((i-1)r+1)F(i),\qquad r\sigma F(n)=\sum_{i=1}^{n}F(i),$$

from which (7) is immediate.

Now suppose that $w=2^i1u\in A(1)^*$. In order to show that $\Gamma_w I(n)$ satisfies the same recurrence (3) as does $N_n(w)$, it suffices to show that for any $F\colon \mathbb{P}\to\mathbb{Z}$,

$$\begin{aligned}(\sigma\tau)^i\sigma F(n)-(\sigma\tau)^i\sigma F(n-1)&=r\sum_{j=1}^{i}(\sigma\tau)^{j-1}\sigma(\sigma\tau)^{i-j}\sigma F(n)\\&\quad+(\sigma\tau)^iF(n)-ir(\sigma\tau)^{i-1}\sigma F(n).\end{aligned}\tag{8}$$

We have

$$\begin{aligned}r\sum_{j=1}^{i}(\sigma\tau)^{j-1}\sigma(\sigma\tau)^{i-j}\sigma&=\sum_{j=1}^{i}(\sigma\tau)^{j-1}(\tau\sigma-\sigma\tau+r\sigma)(\tau\sigma)^{i-j},\qquad\text{by (7)}\\&=\sum_{j=1}^{i}[(\sigma\tau)^{j-1}(\tau\sigma)^{i-j+1}-(\sigma\tau)^j(\tau\sigma)^{i-j}+r(\sigma\tau)^i\sigma]\\&=(\tau\sigma)^i-(\sigma\tau)^i+ir(\sigma\tau)^i\sigma.\end{aligned}\tag{9}$$

But for any $G\colon\mathbb{P}\to\mathbb{Z}$ we have

$$\sigma G(n)-\sigma G(n-1)=G(n).$$

Thus

$$\begin{aligned}(\sigma\tau)^i\sigma F(n)-(\sigma\tau)^i\sigma F(n-1)&=\sigma(\tau\sigma)^iF(n)-\sigma(\tau\sigma)^iF(n-1)\\&=(\tau\sigma)^iF(n).\end{aligned}\tag{10}$$

Hence (8) follows from (9) and (10), as desired.

There remains the case $w=2^i$. We need to show that for any $F\colon\mathbb{P}\to\mathbb{Z}$,

$$(\sigma\tau)^iF(n)-(\sigma\tau)^iF(n-1)=r\sum_{j=1}^{i}(\sigma\tau)^{j-1}\sigma(\sigma\tau)^{i-j}F(n)-(ir-1)(\sigma\tau)^{i-1}F(n).$$

The proof is analogous to that of (8) and will be omitted. □

COROLLARY 2.5. *For all $w\in A(r)^*$, the intervals $[\phi, w]$ in* Fib(r) *and $Z(r)$ have the same number of elements.*

PROOF. Put $n=2$ in Theorem 2.4. □

It would be interesting to find a simple bijective proof of Corollary 2.5. The intervals $[\phi, w]$ in Fib(r) and $Z(r)$ do not in general have the same rank-generating function (e.g. $w=1_i21_j$).

We have the following generalization of Corollary 2.5:

COROLLARY 2.6. *For any $w\in A(r)^*$ and any $j\in\mathbb{P}$, the intervals $[\phi, w]$ in* Fib(r) *and $Z(r)$ have the same number of j-element chains.*

PROOF. For any finite poset P, let $L_n(P)$ be the number of multichains $x_1\leqslant x_2\leqslant\cdots\leqslant x_{n-1}$ of length n in P, and let c_j be the number of j-element chains. Then (see [2, Prop. 3.11.1])

$$L_n(P)=\sum_{j\geqslant1}c_j\binom{n-2}{j-1}.\tag{11}$$

From this it follows easily that the numbers $L_n(P)$ uniquely determine the c_j's. The proof now follows from Theorem 2.4. □

3. A Generalized Rank-generating Function

The *rank-generating function* of a poset P with rank function ρ: $P \to \mathbb{N}$ (defined by $\rho(x)$ = length of longest chain of P with top element x) is given [2, p. 99] by

$$F(P, q) = \sum_{x \in P} q^{\rho(x)}.$$

For Fib(r) and $Z(r)$ we have (see [3, Th. 5.3 and Prop. 5.7])

$$F(\text{Fib}(r), q) = F(Z(r), q) = (1 - rq - q^2)^{-1}. \tag{12}$$

Now, given P as above and $n \in \mathbb{P}$, define

$$F_n(P, q) = \sum_{x_1 \leqslant \cdots \leqslant x_n} q^{\rho(x_n)},$$

summed over all n-element multichains in P. The main result of this section is the following:

THEOREM 3.1. *Let $n \in \mathbb{P}$. Then*

$$F_n(\text{Fib}(r), q) = F_n(Z(r), q) = \prod_{i=1}^{n} (1 - rq - ((i-1)r + 1)q^2)^{-1}.$$

PROOF. It follows from Theorem 2.4 that $F_n(\text{Fib}(r), q) = F_n(Z(r), q)$. We prove Theorem 3.1 for Fib(r) by induction on n. The case $n = 1$ is given by (12). Now assume the result for $n - 1$. Write

$$F_j(\text{Fib}(r), q) = \sum_{t \geqslant 0} f_j(t) q^t. \tag{13}$$

We claim that

$$f_n(t) - f_{n-1}(t) = rf_n(t-1) + ((n-1)r + 1)f_n(t-2), \tag{14}$$

for $n > 0$. (When $n = 0$, (14) is valid for $t \geqslant 3$.)

Now, using the notation of the previous section, we have

$$f_n(t) = \sum_{\rho(v)=t} M_n(v),$$

summed over all words $v \in A(r)^*$ of rank t.

For each $u \in A(r)^*$ of rank $t - 1$ there are r words $v = 1_j u$ (provided that $t \geqslant 1$); while for each $u \in A(r)^*$ of rank $t - 2$ there is one word $v = 2u$ of rank t (provided that $t \geqslant 2$). Hence

$$f_n(t) = r \sum_{\rho(u)=t-1} M_n(1u) + \sum_{\rho(u)=t-2} M_n(2u).$$

By (1) and (2) there follows

$$f_n(t) = r \sum_{\rho(u)=t-1} \sum_{i=1}^{n} M_i(u) + \sum_{\rho(u)=t-2} \sum_{i=1}^{n} ((i-1)r + 1)M_i(u),$$

so (since $n > 0$)

$$\begin{aligned} f_n(t) - f_{n-1}(t) &= r \sum_{\rho(u)=t-1} M_n(u) + \sum_{\rho(u)=t-2} ((n-1)r + 1)M_n(u) \\ &= rf_n(t-1) + ((n-1)r + 1)f_n(t-2), \end{aligned}$$

proving (14).

Now multiply (14) by x^t and sum on $t \geq 0$. This results in (writing $F_j(q)$ for $F_j(\mathrm{Fib}(r), q)$)

$$F_n(q) - F_{n-1}(q) = rqF_n(q) + ((n-1)r+1)q^2F_n(q),$$

for $n > 0$, whence

$$F_n(q) = F_{n-1}(q)/(1 - rq - ((n-1)r+1)q^2).$$

The proof follows by induction. □

Given a graded poset P and $t \in \mathbb{N}$, let

$$P_{[0,t]} = \{x \in P : 0 \leq \rho(x) \leq t\}. \tag{15}$$

In the terminology of [2, Ch. 3.12], $P_{[0,t]}$ is a *rank-selected subposet* of P. Thus, in the notation of (13), $f_n(t)$ is the number of n-element multichains in $\mathrm{Fib}(r)_{[0,t]}$ or $Z(r)_{[0,t]}$, so $f_{n-1}(t)$ (as a function of n) is the zeta polynomial of $\mathrm{Fib}(r)_{[0,t]}$ or $Z(r)_{[0,t]}$. By (11), $f_n(t)$ (or $f_{n-1}(t)$) is a polynomial of degree t and leading coefficient $m_t/t!$, where m_t is the number of maximal chains in $\mathrm{Fib}(r)_{[0,t]}$ or $Z(r)_{[0,t]}$. By [3, Prop. 3.1], we have

$$\sum_{t \geq 0} m_t x^t/t! = \exp(rt + \tfrac{1}{2}rt^2).$$

Equivalently,

$$m_t = \sum_{\pi} r^{c(\pi)}, \tag{16}$$

where π ranges over all involutions in the symmetric group $\mathfrak{S}_t$ and where $c(\pi)$ denotes the number of cycles of π.

We may ask what more can be said about the polynomials $f_n(t)$. By standard properties of rational generating functions [2, Cor. 4.3.1], we have

$$\sum_{n \geq 0} f_n(t)x^n = \frac{W_t(x)}{(1-x)^{t+1}},$$

where for fixed t, $W_t(x)$ is a polynomial in x (called the *$f_n(t)$—Eulerian polynomial*) of degree $\leq t$ with integer coefficients summing to m_t (as defined in (16)). Since $Z(r)$ is a modular lattice (or since $\mathrm{Fib}(r)$ is semimodular), it follows from known results (see [2, Example 3.13.5 and Exercise 3.67(b)]) that $W_t(x)$ has non-negative coefficients. Since $\mathrm{Fib}(1)$ is a distributive lattice, the following combinatorial interpretation of the coefficients of $W_t(x)$ (when $r = 1$) follows easily from the theory of P-partitions [2, Ch. 4.5].

PROPOSITION 3.2. *Given a permutation $\pi \in \mathfrak{S}_t$, write π as a product of disjoint cycles where (a) each cycle is written with its smallest element first, and (b) the cycles are written in increasing order of their smallest element. Let $\tilde{\pi}$ be the permutation (written as a* word) *in $\mathfrak{S}_t$ which results from erasing all parentheses from the above cycle notation. (We may have $\tilde{\pi} = \tilde{\sigma}$ even though $\pi \neq \sigma$; contrast this with the* standard representation *of* [2, p. 17].) *Then, when $r = 1$, we have*

$$W_t(x) = \sum_{\pi} x^{1+d(\tilde{\pi}^{-1})},$$

where π ranges over all involutions *in $\mathfrak{S}_t$, and where $d(\tilde{\pi}^{-1})$ denotes the number of descents* [2, pp. 21–23] *of $(\tilde{\pi})^{-1}$.*

For instance, when $t = 4$ we have the following table:

π	$\tilde{\pi}$	$\tilde{\pi}^{-1}$	$d(\tilde{\pi}^{-1})$
(1)(2)(3)(4)	1234	1234	0
(12)(3)(4)	1234	1234	0
(13)(2)(4)	1324	1324	1
(14)(2)(3)	1423	1342	1
(1)(23)(4)	1234	1234	0
(1)(24)(3)	1243	1243	1
(1)(2)(34)	1234	1234	0
(12)(34)	1234	1234	0
(13)(24)	1324	1324	1
(14)(23)	1423	1342	1

Hence $W_4(x) = 5x + 5x^2$ when $r = 1$. Presumably a similar result holds for any r, but we will not consider this here.

PROPOSITION 3.3. *Fix $r \in \mathbb{P}$. Then the polynomials $W_t(x)$ satisfy the recurrence*

$$W_t(x) = rW_{t-1}(x) + ((rt-1)x - r + 1)W_{t-2}(x) + rx(1-x)W'_{t-2}(x), \qquad t \geq 3, \tag{17}$$

with the initial conditions

$$W_0(x) = 1, \qquad W_1(x) = rx, \qquad W_2(x) = (r-1)x^2 + (r^2+1)x.$$

PROOF. Multiply (14) by x^n and sum on $n \geq 0$. Since (14) is valid for $n \geq 0$ when $t \geq 3$, we obtain for $t \geq 3$ that

$$\frac{W_t(x)}{(1-x)^{t+1}} - \frac{xW_t(x)}{(1-x)^{t+1}} = \frac{rW_{t-1}(x)}{(1-x)^t} + rx\frac{\mathrm{d}}{\mathrm{d}x}\frac{W_{t-2}(x)}{(1-x)^{t-1}} - \frac{(r-1)W_{t-2}(x)}{(1-x)^{t-1}}. \tag{18}$$

When equation (18) is simplified, the recurrence (17) results. It is easy to compute $W_t(x)$ for $0 \leq t \leq 2$ by a direct argument, so the proof follows. □

The values of $W_t(x)$ for $3 \leq t \leq 7$ are given by

$$\begin{aligned}
W_3(x) &= r(3r-2)x^2 + r(r^2+2)x,\\
W_4(x) &= (r-1)(2r-1)x^3 + (6r^3 - 2r^2 + 3r - 2)x^2 + (r^4 + 3r^2 + 1)x,\\
W_5(x) &= r(11r^2 - 12r + 3)x^3 + r(10r^3 + 12r - 6)x^2 + r(r^4 + 4r^2 + 3)x,\\
W_6(x) &= (r-1)(2r-1)(3r-1)x^4 + (35r^4 - 22r^3 + 13r^2 - 12r + 3)x^3\\
&\quad + (15r^5 + 5r^4 + 31r^3 - 8r^2 + 6r - 3)x^2 + (r^6 + 5r^4 + 6r^2 + 1)x,\\
W_7(x) &= 2r(5r-2)(5r^2 - 5r + 1)x^4 + r(85r^4 - 10r^3 + 60r^2 - 60r + 12)x^3\\
&\quad + r(21r^5 + 14r^4 + 65r^3 + 30r - 12)x^2 + r(r^6 + 6r^4 + 10r^2 + 4)x.
\end{aligned}$$

We conclude with a brief discussion of a natural generalization of the polynomials $W_t(x)$. Let P be a graded poset and S a finite subset of $\mathbb{P}$. Generalizing (15), define the *rank-selected poset* [2, p. 131]

$$P_S = \{z \in P\colon \rho(z) \in S\}.$$

Let $\alpha(P, S)$ denote the number of maximal chains of P_S, and define

$$\beta(P, S) = \sum_{T \subseteq S} (-1)^{|S-T|}\, \alpha(P, T).$$

Equivalently,

$$\alpha(P, S) = \sum_{T \subseteq S} \beta(P, T).$$

For more information concerning the numbers $\alpha(P, S)$ and $\beta(P, S)$, see [2, Sect. 3.12–3.13]. In particular [2, Exer. 3.67], we have for $P = \mathrm{Fib}(r)$ and $P = Z(r)$ that

$$W_t(x) = \sum_{S} \beta(P, S) x^{\#(S-\{t\})}, \tag{19}$$

where S ranges over all subsets of $\{1, \ldots, t\}$. Moreover, since $\mathrm{Fib}(r)$ is semimodular and $Z(r)$ is modular, we have [2, Exam. 3.13.5] that $\beta(\mathrm{Fib}(r), S) \geqslant 0$ and $\beta(Z(r), S) \geqslant 0$. However, it is false in general that $\beta(\mathrm{Fib}(r), S) = \beta(Z(r), S)$. For instance,

$$\beta(\mathrm{Fib}(1), \{2, 4\}) = 1, \qquad \beta(Z(1), \{2, 4\}) = 2.$$

The techniques of [2, Sect. 3.12] lead to the following result, which together with (19) imply Proposition 3.2 by an easy argument (so that Proposition 3.4 may be regarded as a generalization of Proposition 3.2).

Proposition 3.4. *Let S be a finite subset of $\mathbb{P}$. Then $\beta(\mathrm{Fib}(1), S)$ is equal to the number of permutations $\pi = (a_1, a_2, a_3, \ldots)$ of $\mathbb{P}$ satisfying:*
(a) $a_i = i$ for all but finitely many i;
(b) $2i$ and $2i + 1$ appear to the right of $2i - 1$ for all $i \in \mathbb{P}$;
(c) $D(\pi) = S$, where $D(\pi)$ denotes the descent set of π [2, p. 21].

It would be interesting to find a similar result for $\mathrm{Fib}(r)$ when $r \geqslant 2$ and for $Z(r)$ when $r \geqslant 1$.

Acknowledgment

This work was partially supported by NSF grant #DMS 8401376.

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Received 6 March 1989 and accepted 17 August 1989

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